Groups and Subgroups, Cosets

Time : 1 Hour **Maximum Marks: 40**

Compulsory Questions

Q1. Show that the set S = {1, 2, 3, 4, 5} is not a group under the composition 'multiplication modulo 6'.

Q2. If every element of a group is its own inverse, then show that the group is abelian.

Q3. Find the order of each element of the multiplicative group (1, –1, i, –i).

Q4. Let (G, .) be a group such that a5 = e and a . b. a–1 = b2 for all a, b $\in$ G, find *o*(b).

Q5. If H_1 and H_2 are two subgroup of G, then $H_1 \cap H_2$ is also a subgroup of G.

Q6. Let G be a group with binary operation denoted as multiplication. The set $H = \{h \in G :: hx = xh \text{ for all } x \in G\}$ is called the centre of the group G. Show that the centre of G is a subgroup of G.

Q7. Every cyclic group is an abelian group.

Q8. Any two right (left) cosets of a subgroup are either disjoint or identical.

Q9. If H is a subgroup of G, then prove that there is a one to one correspondence between any two right cosets of H in G.

Q10. If G is the additive group of intergers and H is the subgroup of G obtained on multiplying the elements of G by 4, find the index of H in G.

Some Important Question

Q1. Let R be the set of all real numbers other than –1 and '*' be the binary operation on R defined by a * b = a + b + ab. Determine the identity element and inverse of *a*.

Q2. If a group has four elements, show that it must be abelian.

Q3. A necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that $a \in H, b \in H \Rightarrow ab^{-1} \in H$, where b^{-1} is the inverse of b in G.

Q4. Every finite group of prime order is cyclic.

Q5. Every subgroup of a cyclic group is cyclic.

Q6. Show that if G is an infinite cyclic group, then G has exactly two generators.

Q7. If an abelian group of order 6 contains an element of order 3, show that it must be a cyclic groups.

Q8. Let *a* and *b* be two elements of finite order of a group G. If o(a) and o(b) are co-prime and ab = ba, prove that o(ab) = o(a) . o(b).

Q9. The order of each subgroup of a finite group is a divisor of the order of the group.

Q10. The order of every element of a finite group is a divisor of the order of the group

Homomorphisms and Automorphism, Permutation Groups

Time : 1 Hour **Maximum Marks: 40**

Compulsory Questions

Q1. Let G be the additive group of real numbers and G' be the multiplicative group of positive real numbers. If $\phi : G \to G'$ is a mapping defined by $f(x) = e^x$ for all $x \in G$, then show that ϕ is an isomorphism of G onto G′.

Q2. If $f : G \to G'$ is a homomorphism then prove that kernel of *f* is a normal subgroup of G.

Q3. Let G be an abelian group and $f : G \to G$ be such that $f(x) = x^{-1}$. Show that *f* is an automorphism.

Q4. If a group G has a non-trivial automorphism, then it has atleast three elements.

Q5. Let $f : G \to G$ be a homomorphism. Let *f* commutes with every inner automorphism of G. Show that $H = \{x \in G : f^2(x) = f(x)\}$ is a normal subgroup of G.

Q6. Show that the group of automorphisms of a finite cyclic group is abelian.

Q7. Let Z (G) be the centre of a group G. If G/Z is cyclic, then prove that G is abelian.

Q8. The normalizer of a ∈ G is a subgroup of G.

Q9. Write all elements of symmetric group S_3 as product of disjoint cycles.

Q10. Find the centre of permutation group S_3.

Some Important Question

Q1. The necessary and sufficient condition for a homomorphism f to be one one-is that kernel *f* ={*e*}, where *e* is identity of domain.

Q2. Every homomorphic image of a group G is isomorphic to some quotient groups of G.

Q3. If G is a finite abelian group of order n and m is a positive integer such that (m ,n) = 1, then show that $: G \to G$ defined by $f(x) = x^m$ is an automorphism.

Q4. Let 'a' be a fixed element of a group G. Then the mapping $T_a : G \to G$, such that $T_a(x) = a^{-1}xa$ is an automorphism of a group is automorphism of that group.

Q5. The set Inn (G) of all inner automorphisms of a group G is a normal subgroup of the group Aut (G) of its automorphisms.

Q6. The set of all element of a group G which commute with every element of the group is a normal subgroup of G.

Q7. The set Inn (G) of all inner automorphism of a group G is isomorphic to the quotient group G / Z(G), where Z (G) is the centre of G, i.e., Inn (G) ≅ G/Z (G)

Q8. If p is a prime number and G is a non-abelian group0 of order p^3, show that Z(G) has exactly p elements.

Q9. Every permutation can be expressed as the product of disjoint cycles.

Q10. Every group is isomorphic to a permutation group.

Rings and Fields, Ideals and Quotient Rings

Time : 1 Hour **Maximum Marks: 40**

Compulsory Questions

Q1. A division rings (skew field) has a no zero divisors.

Q2. Give an example of a division ring (skew field) which is not a field.

Q3. The intersection of two subrings is a ring.

Q4. Show that the set of matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ is subring of the ring of 2 × 2 matrices with integral elements.

Q5. S_1 and S_2 be two ideals of a ring R. Let $S_1 + S_2 = \{a_1 + a_2 : a_1 \in S_1, a_2 \in S_2\}$. Then $S_1 + S_2$ is called sum of ideals S_1 and S_2 and is an ideal of R.

Q6. A division ring is a simple ring.

Q7. A field has no proper ideals.

Q8. Let R be a ring of 2 × 2 matrices over integers. Let $S = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a, b \text{ integers} \right\}$. Then S is a left ideal but not right ideal.

Q9. Let R/S be a quotient ring. Prove that (i) if R is commutative, then so is R/S (ii) if R has a unity element 1, then S+1 is a unity element of R/S.

Q10. Let R be a commutative ring and S is an ideal of R. Then R/S is an integral domain iff S is a prime ideal

Some Important Question

Q1. Every field is an integral domain.

Q2. Every finite non-zero integral domain is a field.

Q3. The necessary and sufficient conditions for a non-empty subset S of the ring R to be a subring of R are: (i) $a, b \in S \Rightarrow a - b \in S$ (ii) $a, b \in S \Rightarrow a \,.\, b \in S$

Q4. Prove that characteristic of an integral domain is either zero or a prime number.

Q5. If S_1, S_2 are two ideals of a ring R, then their product S_1S_2 is an ideal of R.

Q6. Let R be a ring with unity element such that R has no right ideals except {0} and R. Prove that R is a division ring.

Q7. The ring of integrers is a principal ideal ring.

Q8. An ideal of a ring of integers is a maximal iff it is generated by some prime integer.

Q9. Let R be a ring and S, an ideal of R. Then R/S is ring under the addition and multiplication defined as under:

$(S+a)+(S+b) = S+(a+b)$ and $(S+a)(S+b) = S+ab$ for $S+a, S+b \in R/S$

Q10. An ideal S of a commutative ring R with unity is maximal iff R/S is a field.

Euclidean Rings

Time : 1 Hour **Maximum Marks: 40**

Compulsory Questions

Q1. If a is a unit of commutative ring R with unity element, then a^{-1} is also a unit in R.

Q2. The integral domain < Z, x, . > of integers is an Euclidean domain.

Q3. Prove that every field is an Euclidean ring.

Q4. Show that an element a in an Euclidean ring R is a unit iff d(a) = d(1).

Q5. Let a, b, c be arbitrary element of an Euclidean ring R. If (a, b) = 1 and a / be, then a/c.

Q6. Fill all the units of $Z\left[\sqrt{-5}\right]$.

Q7. Illustrate with the help of an example that there exits two elements a, b in an Euclidean domain such that d(a) = d(b), but a, b are are not associates.

Q8. Let R be a Euclidean ring and a, b be non-zero elements of R. If a and b are associates, show that d(a) = d(b).

Q9. Show that the units of Z(i) are ± 1, ± i.

Q10. Show that 1 + i is an irreducible element in the ring Z[i] of Gaussian integers.

Some Important Question

Q1. The product of two units a, b ∈ R is also a unit of R.

Q2. The ring of Gaussian integers is an Euclidean domain (ring).

Q3. Every Euclidean ring is a principal ideal ring.

Q4. Let R be an Euclidean ring. Then any two elements a and b in R have a greatest common divisor.

Q5. If R is a principal ideal domain, then any two non-zero element a, b ∈ R have a *l.c.m.*

Q6. An ideal S of an Euclidean ring R is maximum iff S is generated by some prime element of R.

Q7. An element in a principal ideal domain is prime element iff it is irreducible.

Q8. Let R be a principal ideal domain which is not a field. Then an ideal S = < a_0> is a miximum ideal iff a_0 is an irreducible element.

Q9. Show that $\sqrt{-5}$ is a prime element of the ring $Z\sqrt{-5} = \{a + \sqrt{-5}b : a, b \in Z\}$.

Q10. Show that every non-zero prime ideal of a principal ideal domain is maximal.